

EQUIVARIANT MINIMAL MODELS

BY

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ABSTRACT. Let G be a finite group. We give an algebraicization of rational G -homotopy theory analogous to Sullivan's theory of minimal models in ordinary homotopy theory.

1. Introduction. We consider simplicial complexes X on which a finite group G acts simplicially. We assume that $X^g = \{x \in X \mid gx = x\}$ is a simplicial complex for each $g \in G$; this can always be arranged by repeated triangulation. We assume once and for all that all G -spaces in sight are G -simple in the sense that the fixed point spaces X^H are simple for all subgroups H of G . We also assume that the rational cohomology of each X^H is of finite type.

DEFINITION 1.1. A G -map $f: X \rightarrow Y$ is a rational G -homotopy equivalence if each $f^H: X^H \rightarrow Y^H$ induces an isomorphism on rational cohomology.

DEFINITION 1.2. Two G -spaces X and Y have the same rational G -homotopy type if there is a chain of rational G -homotopy equivalences connecting them.

These definitions are motivated by the following equivariant version of the Whitehead theorem [1]:

THEOREM 1.3 (G. BREDON). *An equivariant map $f: X \rightarrow Y$ between two G -complexes is an equivariant homotopy equivalence iff $(f^H)_*: \pi_*(X^H) \rightarrow \pi_*(Y^H)$ is an isomorphism for each subgroup $H \subseteq G$.*

By using the same equivariant obstruction theory which leads to this result, we constructed in [5] an equivariant Postnikov decomposition of a G -complex X , i.e., a decomposition of X in a sequence of G -fibrations

$$\{pt\} \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots,$$

the fibers F_n of which have the property that F_n^H is an Eilenberg-Mac Lane space of dimension n for each $H \subseteq G$ (we can speak of a G -action on the fiber since X has at least one fixed point). We shall review this material in §2. Our goal is to construct an equivariant minimal model for X which is the algebraic analogue of the rationalization (localization at 0) of the equivariant Postnikov decomposition of X .

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Sullivan's minimal model of a simplicial complex X is a differential graded \mathbb{Q} -algebra (abbreviated DGA), \mathfrak{M}_X which is free and has decomposable differential and which maps by a cohomology isomorphism into the de Rham algebra \mathfrak{S}_X of differential forms on X . In the case of a G -complex, we consider not only the DGA \mathfrak{S}_X with the induced G -action on it, but also the de Rham algebras \mathfrak{S}_{X^H} of all the fixed point sets X^H and the canonical maps relating them. This means that we consider a functor $\underline{\mathfrak{S}}_X$ of a particular sort.

Let \mathcal{O}_G be the *category of canonical orbits*. Its objects are the orbits G/H and its morphisms are the G -maps between them. We define a (contravariant) *rational coefficient system for G* to be a contravariant functor from \mathcal{O}_G to the category of rational vector spaces. Let Vec_G denote the category of rational coefficient systems and let Vec_G^* denote the dual category of covariant functors from \mathcal{O}_G to rational vector spaces. An understanding of the *injective* objects of the abelian category Vec_G^* turns out to be central to our work, and we analyze the projective objects of Vec_G in §3.

Define a *system of DGA's* to be a covariant functor \mathfrak{A} from \mathcal{O}_G to the category of (simple) DGA's which is injective when regarded as (graded) object of Vec_G^* by neglect of structure.

The functor $\underline{\mathfrak{S}}_X$ above is a system of DGA's. We shall prove this in §4.

In §5 we shall specify a notion of minimal system of DGA's. This will allow us to prove the following result, which is the technical heart of our work, in §7:

THEOREM 1.4. *For each system of DGA's \mathfrak{A} , there is a minimal system of DGA's \mathfrak{M} and a map of systems of DGA's $\rho: \mathfrak{M} \rightarrow \mathfrak{A}$ which induces an isomorphism on cohomology (at each G/H).*

We call \mathfrak{M} the *minimal model* of \mathfrak{A} . It is unique up to isomorphism. We then define the minimal model \mathfrak{M}_X of X to be the minimal model of $\underline{\mathfrak{S}}_X$. With this definitional apparatus, we prove the following theorem in §6:

THEOREM 1.5. *The correspondence $X \mapsto \mathfrak{M}_X$ induces a bijection between equivariant rational homotopy types of G -spaces on the one hand and isomorphism classes of minimal systems of DGA's on the other.*

Say that a G -complex Y is rational if each homotopy group $\pi_n(Y^H)$ is a rational vector space. We shall define the notion of homotopy between morphisms of systems of DGA's in §4 and we shall prove the following theorem in §6:

THEOREM 1.6. *If X is a G -complex and Y a rational G -complex, there is a bijection*

$$[X, Y]_G \cong [\mathfrak{M}_Y, \mathfrak{M}_X].$$

This algebraicizes rational G -homotopy classes of G -maps.

Of course, this work presupposes the appropriate notion of cohomology which is that introduced by Bredon [1] and recalled in §2. In a sense to be made precise, the minimal model \mathfrak{M}_X computes the rational Bredon cohomology of X .

REMARK ON NOTATION. Throughout this paper G stands for a fixed finite group, $H \subseteq G$ means H is a subgroup of G and $H \subset G$ means H is a proper subgroup of G .

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2. The equivariant Postnikov decomposition and its rationalization. In this section we give a summary of Bredon's equivariant cohomology and obstruction theory [1] and of our construction in [5] of the equivariant Postnikov decomposition and its rationalization.

We first describe the morphisms in the category of canonical orbits \mathcal{O}_G . Any morphism, i.e., any equivariant map $G/H \rightarrow G/K$, is of the form $gH \xrightarrow{\hat{a}} gaH$ for some $a \in G$ with the property $a^{-1}Ha \subseteq K$. Moreover, $\hat{a} = \hat{b}$ iff $ab^{-1} \in K$. It can be also easily checked that

$$E(G/H, G/H) \cong N(H)/H \cong (G/H)^H$$

where the first set is the set of G -maps $G/H \rightarrow G/H$ and $N(H)$ is the normalizer of H . The bijections are given by $N(H) \ni a \mapsto aH \in (G/H)^H$ and $a \mapsto \hat{a} \in E(G/H, G/H)$, $\hat{a}(gH) = gaH$. In general $E(G/K, G/H) \cong (G/H)^K$.

A coefficient system for G is a contravariant functor from \mathcal{O}_G to the category of abelian groups.

EXAMPLES. (1) If A is a $\mathbf{Z}[G]$ -module, define a coefficient system \underline{A} by $\underline{A}(G/H) = A^H$, with $\underline{A}(\hat{a}): A^K \rightarrow A^H$ specified by $\underline{A}(\hat{a})(x) = ax$. Note that the multiplication $a: A \rightarrow A$ maps A^K into A^H since $x \in A^K$ implies $hax = aka^{-1}ax = akx = ax$, where $h \in H$ and $k \in K$.

(2) If X is a G -complex, define coefficient systems $\underline{\pi}_*(X)$, $\underline{H}_*(X)$ and $\underline{C}_*(X)$ by

$$\underline{\pi}_*(X)(G/H) = \pi_*(X^H), \quad \underline{H}_*(X)(G/H) = H_*(X^H; \mathbf{Z})$$

and

$$\underline{C}_*(X)(G/H) = C_*(X^H)$$

on objects, where $C_n(X) = H_n(X^{(n)}, X^{(n-1)}; \mathbf{Z})$. A morphism $\hat{a}: G/H \rightarrow G/K$ induces a map $a: X^K \rightarrow X^H$ and therefore a map on homotopy, homology and cell complex.

Let M be any coefficient system for G . Since the category of such systems is abelian (see [4, p. 258]), the sets $\text{Hom}(\underline{C}_n(X), M)$ are abelian groups. Moreover, they form a cochain complex with coboundary operator induced by the boundary operator of $\underline{C}_*(X)$. The equivariant cohomology of the G -complex X is defined by

$$(2.1) \quad H_G^n(X; M) = H^n(\text{Hom}(\underline{C}_*(X), M)).$$

The relation between $H_G^n(X; M)$ and $\underline{H}_*(X)$ is given by a spectral sequence. The category of coefficient systems for G has enough injectives, hence we can construct an injective resolution $M^{(*)}$ of M . The coefficient system $\underline{C}_*(X)$ is a *projective* object in this category. We consider the double complex $\text{Hom}(\underline{C}_*(X), M^{(*)})$. Standard homological algebra applied to this complex yields a spectral sequence

$$(2.2) \quad E_2^{s,t} = \text{Ext}^s(\underline{H}_t(X), M) \Rightarrow H_G^{s+t}(X; M).$$

The relative equivariant cohomology $H_G^n(X, Y; M)$ is also defined and there is a long exact sequence

$$\cdots \rightarrow H_G^n(X, Y; M) \rightarrow H_G^n(X; M) \rightarrow H_G^n(Y; M) \rightarrow H_G^{n+1}(X, Y; M) \rightarrow \cdots.$$

This is the right cohomology theory in which to express obstructions for extending or lifting G -maps.

(2.3) *Equivariant obstruction theory.* Let Y be a G -subcomplex of X and $\varphi: X^{(n)} \cup Y \rightarrow Z$ be a G -map. Let σ be an $(n+1)$ -simplex of X and $G_\sigma \subseteq G$ its isotropy group. It is clear that $\varphi(S^n) \subseteq Z^{G_\sigma}$, where S^n is the boundary of σ , i.e., $\varphi|S^n$ defines an element in $\pi_n(Z^{G_\sigma})$. Hence φ defines a natural transformation $c_\varphi \in \text{Hom}(\underline{C}_{n+1}(X, Y); \pi_n(Z))$. This is a cocycle and defines a cohomology class

$$[c_\varphi] \in H_G^{n+1}(X, Y; \pi_n(Z)),$$

which is the obstruction to extending $\varphi|X^{(n-1)} \cup Y$ to a G -map $X^{(n+1)} \cup Y \rightarrow Z$.

Let M be a coefficient system for G . An *Eilenberg-Mac Lane G -complex* of type (M, n) is a G -complex K with the property $\pi_n(K) = M$ and $\pi_m(K) = 0$ for $m \neq n$. In other words, K^H is an Eilenberg-Mac Lane space of dimension n for every $H \subseteq G$. Such G -complexes exist and fulfill the relation

$$[X, K]_G \cong H_G^n(X; M).$$

We may indicate one way an Eilenberg-Mac Lane G -complex can be constructed because of the analogy between this construction and an algebraic construction in §5. First, observe that there are enough projectives in the category of coefficient systems. One can construct a projective coefficient system as follows. Let T be a G -set and let $\underline{F}(T)$ be defined by $\underline{F}(T)(G/H) = F(T^H)$, where $F(T^H)$ is the free abelian group generated by T^H (note that $F(T^H) \neq F(T)^H$). For instance, $\underline{C}_n(X)$ is a projective coefficient system of this form; here T consists of the n -cells of X . In order to construct an Eilenberg-Mac Lane G -complex $K(M, n)$, first consider a projective resolution of M

$$\cdots \rightarrow \underline{F}(T_1) \xrightarrow{f_1} \underline{F}(T_0) \rightarrow M \rightarrow 0.$$

Let $S^n T_0^+$ be the one point union of n -spheres, one for each element of T_0 , and let $CS^n T_1^+$ be the one point union of $(n+1)$ -cells, one for each element of T_1 . In analogy with the construction of ordinary Eilenberg-Mac Lane complexes, the G -complex $S^n T_0^+ \cup_{f_1} CS^n T_1^+$ has the right homotopy groups in dimension n . The higher homotopy groups are killed by attaching cells similarly.

An equivariant map $p: E \rightarrow X$ is a G -fibration if it has the equivariant homotopy lifting property. Let $e_0 \in E^G$ be a base point and put $x_0 = p(e_0)$. The G -space $F = p^{-1}(x_0)$ is called the fiber of p . Let Y be a G -complex, $f: X \rightarrow Y$ a G -map and $\tilde{f}: Y^{(n)} \rightarrow E$ an equivariant lifting of $f|Y^{(n)}$, i.e., $\rho \circ \tilde{f} = f|Y^{(n)}$. There is a cohomology class

$$[c_{(f, \tilde{f})}] \in H_G^{n+1}(Y; \pi_n(F))$$

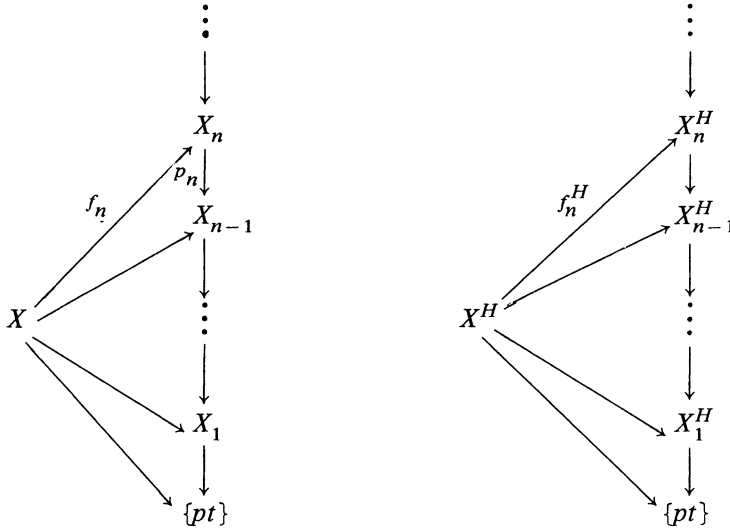
which vanishes iff the lifting $\tilde{f}|Y^{(n-1)}$ can be extended equivariantly to a lifting $Y^{(n+1)} \rightarrow E$. Let H be a G -homotopy between two such liftings $\tilde{f}|Y^{(n)}$ and $\tilde{f}'|Y^{(n)}$.

The obstruction to extending $H|Y^{(n-1)}$ to a G -homotopy $Y^{(n+1)} \times I \rightarrow E$ lies in $H_G^n(Y; \pi_n(F))$.

We refer to [1] for details of the material so far.

(2.4) A *classification theorem* follows easily from the obstruction theory above. Let $p: E \rightarrow X$ be a G -fibration, the fiber of which is an Eilenberg-Mac Lane G -complex $K(M, n)$. Let $f: Y \rightarrow X$ be a G -map and $\tilde{f}: Y \rightarrow E$ a fixed equivariant lifting of f . Then there is a bijection between the set of equivariant homotopy classes of equivariant liftings of f on the one hand and $H_G^n(Y; M)$ on the other. A class $[\tilde{f}']$ corresponds to the first obstruction to the existence of a G -homotopy between \tilde{f} and \tilde{f}' .

(2.5) The *equivariant Postnikov decomposition* of a G -space X is a chain of G -spaces and G -maps such that the first diagram below commutes and the second is the ordinary Postnikov decomposition of X^H for every $H \subseteq G$. We insist that $p_n: X_n \rightarrow X_{n-1}$ be a G -fibration. Then the fiber is obviously an Eilenberg-Mac Lane G -complex $K(\pi_n(X), n)$.



A G -fibration, the fiber of which is an Eilenberg-Mac Lane G -complex $K(M, n)$, is called a *principal G -fibration*. G -equivalence classes of principal G -fibrations over X_{n-1} are classified by $H_G^{n+1}(X_{n-1}; M) \cong [X_{n-1}, K(M, n+1)]_G$. Given a cohomology class in $H_G^{n+1}(X_{n-1}; M)$, i.e., a G -map $f: X_{n-1} \rightarrow K(M, n+1)$, we can construct a G -fibration over X_{n-1} by considering the induced fibration by f of the path fibration

$$\Omega K(M, n+1) \rightarrow PK(M, n+1) \rightarrow K(M, n+1).$$

So the G -fibration $p_n: X_n \rightarrow X_{n-1}$ is characterized by a cohomology class $k^{n+1} \in H_G^{n+1}(X_{n-1}; \pi_n(X))$, the $(n+1)$ st *equivariant k -invariant* of X .

The equivariant k -invariants of a G -space X are determined by X in the following way: By using the spectral sequence above we can prove

$$H_G^{n+1}(X_{n-1}, X; \pi_n(X)) \cong \text{Hom}(\pi_n(X), \pi_n(X)).$$

Then the cohomology class ι_n corresponding to $\text{id} \in \text{Hom}(\pi_n(X), \pi_n(X))$ is mapped by

$$H_G^{n+1}(X_{n-1}, X; \pi_n(X)) \rightarrow H_G^{n+1}(X_{n-1}; \pi_n(X))$$

to k^{n+1} . The G -Postnikov decomposition is unique up to G -homotopy equivalence. Moreover, by passage to inverse limits, the G -Postnikov decomposition determines a G -space whose semisimplicial resolution (after barycentric subdivision) is a G -complex G -homotopy equivalent to X .

(2.6) *Equivariant rationalization.* A G -space X is said to be *rational* (a \mathbf{Q} - G -space) if the homotopy groups $\pi_i(X^H)$ are \mathbf{Q} -vector spaces for each $H \subseteq G$. For each G -space X there exist a \mathbf{Q} - G -space X_0 and a G -map $f: X \rightarrow X_0$ such that the following equivalent statements hold:

(1) $f^*: H_G^*(X; M \otimes \mathbf{Q}) \rightarrow H_G^*(X_0; M \otimes \mathbf{Q})$ is an isomorphism for all coefficient systems M ;

(2) $f_*: \underline{H}_*(X; \mathbf{Q}) \rightarrow \underline{H}_*(X_0; \mathbf{Q}) \cong \underline{H}_*(X_0; \mathbf{Z})$ is an isomorphism;

(3) $f_* \otimes \text{id}: \pi_*(X) \otimes \mathbf{Q} \rightarrow \pi_*(X_0) \otimes \mathbf{Q} \cong \pi_*(X_0)$ is an isomorphism;

(4) the map f has the universal lifting property, i.e., every G -map $g: X \rightarrow Y$, where Y is a \mathbf{Q} - G -space, factors through X_0 , uniquely up to G -homotopy,

$$\begin{array}{ccc} & & X_0 \\ & \nearrow f & \vdots \\ X & & Y \\ & \searrow g & \vdots \end{array}$$

The existence of a G -rationalization of X is proved by inductively constructing G -rationalizations φ_n of the G -spaces X_n of the G -Postnikov decomposition. We first localize the Eilenberg-Mac Lane G -complexes by constructing

$$K(\pi_n(X), n+1) \xrightarrow{i} K(\pi_n(X) \otimes \mathbf{Q}, n+1)$$

and then consider the “localized” k -invariants k_0^{n+1} :

$$\begin{array}{ccccc} & & K(\pi_n(X), n+1) & \xrightarrow{i} & K(\pi_n(X) \otimes \mathbf{Q}, n+1) \\ & \nearrow k^{n+1} & & & \nearrow k_0^{n+1} \\ X_{n-1} & \xrightarrow{\varphi_{n-1}} & X_{n-1,0} & & \end{array}$$

We can say, equivalently to Definition 1.2, that the G -rational homotopy type of a G -space X corresponds bijectively to the G -homotopy type of its rationalization X_0 .

3. Projective rational coefficient systems. In this section we shall study the projective (contravariant) rational coefficient systems and give a complete description of projective covers.

The category Vec_G of rational coefficient systems is abelian and a morphism $f: M \rightarrow N$ is an epimorphism (respectively monomorphism) iff $f(G/H): M(G/H) \rightarrow N(G/H)$ is an epimorphism (respectively monomorphism) of vector spaces. The examples of coefficient systems given in §2 can obviously be tensored over \mathbb{Q} to give rational coefficient systems. As before, given a G -set T , we can define a projective rational coefficient system $\underline{\mathbb{Q}}(T)$ by $\underline{\mathbb{Q}}(T)(G/H) = \mathbb{Q}(T^H)$, where $\mathbb{Q}(T^H)$ is the vector space over \mathbb{Q} with basis T^H . We shall describe all projectives in Vec_G .

We first make some easily checked elementary remarks. The set G/H can be written as disjoint union

$$G/H = (G/H)^H \cup (G/H)^{H_1} \cup \cdots \cup (G/H)^{H_m},$$

where $H = H_0, H_1, \dots, H_m$ are the distinct subgroups of G conjugate to H . The sets $(G/H)^{H_i}$ are isomorphic, and $N(H)/H$ acts freely and transitively on each from the right by $(gH)a = (ga)H$. The action of G on G/H by left multiplication permutes the copies $(G/H)^{H_i}$, and $(G/H)^K$ contains those copies $(G/H)^{H_i}$ for which $K \subseteq H_i$. Thus $N(H)/H$ acts on $(G/H)^K$ from the right. The vector space $\mathbb{Q}((G/H)^K)$ is a free right $\mathbb{Q}(N(H)/H)$ -module, a basis of which is any set of elements of $(G/H)^K$ such that exactly one element belongs to each copy $(G/H)^{H_i} \subseteq (G/H)^K$. Therefore, if V_H is a left $\mathbb{Q}(N(H)/H)$ -module, the tensor product $(\mathbb{Q}(G/H)^K) \otimes_{\mathbb{Q}(N(H)/H)} V_H$ is equal to a direct sum of copies of V_H . Of course, V_H is necessarily projective since the group ring $\mathbb{Q}(N(H)/H)$ is semisimple.

DEFINITION 3.1. Define a rational coefficient system \underline{V}_H by

$$\underline{V}_H(G/K) = \mathbb{Q}((G/H)^K) \otimes_{\mathbb{Q}(N(H)/H)} V_H$$

on the objects of \mathcal{O}_G . Given a morphism $\hat{a} \in E(G/K, G/K')$, define $\underline{V}_H(\hat{a}): \underline{V}_H(G/K') \rightarrow \underline{V}_H(G/K)$ by

$$\underline{V}_H(\hat{a})(\hat{b} \otimes v) = (\hat{b} \circ \hat{a}) \otimes v,$$

where $\hat{b} \in (G/H)^{K'} = E(G/K', G/H)$ and $v \in V_H$.

PROPOSITION 3.2. \underline{V}_H is projective in the category Vec_G .

PROOF. Let $M \xrightarrow{f} N \rightarrow 0$ be an exact sequence in Vec_G and $g: \underline{V}_H \rightarrow M$ a morphism. We have to construct a morphism $h: \underline{V}_H \rightarrow M$ such that the diagram

$$\begin{array}{ccccc} & & \underline{V}_H & & \\ & \nearrow h & \downarrow g & & \\ M & \xrightarrow{f} & N & \rightarrow & 0 \end{array}$$

commutes. Since $\underline{V}_H(G/H) = \mathbf{Q}(N(H)/H) \otimes_{\mathbf{Q}(N(H)/H)} V_H = V_H$, there exists an $N(H)/H$ -map $h(G/H): V_H \rightarrow M(G/H)$ such that the diagram

$$\begin{array}{ccc} & & V_H \\ & \nearrow h(G/H) & \downarrow g(G/H) \\ M(G/H) & \xrightarrow{f(G/H)} & N(G/H) \rightarrow 0 \end{array}$$

commutes. We define $h(G/K): \underline{V}_H(G/K) \rightarrow M(G/K)$ by

$$h(G/K)(\hat{a} \otimes v) = M(\hat{a})(h(G/H)(v)),$$

where $v \in V_H$ and $\hat{a} \in (G/H)^K = E(G/K, G/H)$.

COROLLARY 3.3. $\mathbf{Q}(T)$ is projective in Vec_G .

PROOF. The G -set T is the disjoint union of its orbits. Therefore, it suffices to prove that $\mathbf{Q}(G/H)$ is projective. But $\mathbf{Q}(G/H)$ is of the form \underline{V}_H , where $V_H = \mathbf{Q}(N(H)/H)$.

PROPOSITION 3.4. Every projective object P in Vec_G can be written as $\bigoplus_{H \subseteq G} \underline{V}_H$.

PROOF. In each conjugacy class (H) of subgroups of G , we consider just one subgroup H . The subspace $\sum_{H' \supset H} P(\hat{e}_{H,H'})(P(G/H'))$ of $P(G/H)$ is a $\mathbf{Q}(N(H)/H)$ -submodule, where $\hat{e}_{H,H'}: G/H \rightarrow G/H'$ is the projection. Let $V_H \subseteq P(G/H)$ be a $\mathbf{Q}(N(H)/H)$ -submodule such that

$$V_H \oplus \sum_{H' \supset H} P(\hat{e}_{H,H'})(P(G/H')) = P(G/H).$$

We define \underline{V}_H as above and $\underline{V}_{H_i} = 0$ for any other subgroup conjugate to H . Consider the morphism $p: \bigoplus_{H \subseteq G} \underline{V}_H \rightarrow P$ defined by $p(G/K)(\hat{a} \otimes v) = P(\hat{a})(v)$, where $\hat{a} \in (G/H)^K$ and $v \in V_H \subseteq P(G/H)$. Clearly p is an epimorphism. Since P is projective, there exists $p': P \rightarrow \bigoplus_{H \subseteq G} \underline{V}_H$ such that $p \circ p' = \text{id}$. We shall show that p is also a monomorphism by showing that p' is an epimorphism. It is clear that $p'(G/G)$ is surjective. By using an inductive argument, it is easy to see that $p'(G/H)$ is surjective after having assumed that $p'(G/H')$ is surjective for every $H' \supset H$.

DEFINITION 3.5. A *projective cover* of a rational coefficient system M is an epimorphism $p: P \rightarrow M$, where P is projective in Vec_G , with the following property: For any projective P' in Vec_G and any epimorphism $p': P' \rightarrow M$ there exists an epimorphism $q: P' \rightarrow P$ such that the diagram

$$\begin{array}{ccc} & P' & \\ q \nearrow & & \searrow p' \\ P & \xrightarrow{p} & M \end{array}$$

commutes.

PROPOSITION 3.6. Any object M in Vec_G has a projective cover which is unique up to isomorphism.

PROOF. We construct $P = \bigoplus_{H \subseteq G} \underline{V}_H$ and $p: \bigoplus_{H \subseteq G} \underline{V}_H \rightarrow M$ as in the previous proposition. The space V_H is the equivariant complement of

$\Sigma_{H' \supset H} M(\hat{e}_{H,H'})(M(G/H')) \subseteq M(G/H)$ or equal to 0. Let P' be a projective in Vec_G and $p': P' \rightarrow M$ an epimorphism. Since P' is projective there is a morphism $q: P' \rightarrow P$ such that the diagram

$$\begin{array}{ccc} & & P' \\ & \nearrow q & \downarrow p' \\ P & \xrightarrow{p} & M \end{array}$$

commutes. According to Proposition 3.4, we can write $P' = \bigoplus_{H \subseteq G} W_H$. Moreover, we can choose W_H to be a subspace of $p'^{-1}(G/H)(V_H)$. Then $q(G/H)(W_H) = V_H$. Obviously $q(G/H)$ is surjective for every subgroup H . This completes the proof.

4. Systems of DGA's and the equivariant de Rham theorem. In this section we set up the right algebraic framework for the study of equivariant minimal models.

Recall that Vec_G^* is the category of covariant functors from \mathcal{O}_G to the category of rational vector spaces, or *covariant rational coefficient systems for G* . Vec_G^* is dual to Vec_G . According to the results of the previous section, the *injective* objects in Vec_G^* are of the form $\bigoplus_{H \subseteq G} V_H^*$; here V_H^* is defined by

$$(4.1) \quad V_H^*(G/K) = \text{Hom}_{\mathbf{Q}(N(H)/H)}(\mathbf{Q}((G/H)^K), V_H^*),$$

where V_H^* is a right $\mathbf{Q}(N(H)/H)$ -module. Moreover, any object M in Vec_G^* has an injective envelope I , defined dually to the notion of a projective cover. In this case, $I = \bigoplus_{H \subseteq G} V_H^*$, where $V_H^* \subseteq M(G/H)$ is defined by

$$V_H^* = \bigcap_{H' \supset H} \ker M(\hat{e}_{H,H'});$$

$\hat{e}_{H,H'}: G/H \rightarrow G/H'$ is the projection.

The most important example of an injective object in Vec_G^* is $\underline{\mathcal{E}}_X$, where X is a G -complex, defined in the following way:

On a simplicial complex X , one can consider analogues of differential forms, the so-called *PL forms*, which were first introduced by H. Whitney [6]. Let

$$\sigma^n \cong \Delta^n = \left\{ (t_0, t_1, \dots, t_n) \mid 0 \leq t_i \leq 1, \sum_{i=0}^n t_i = 1 \right\}$$

be an n -simplex of X canonically embedded in R^{n+1} . A *PL form* of degree p on σ^n is a polynomial form

$$\sum_I f_I(t_0, \dots, t_n) dt_{i_1} \wedge \dots \wedge dt_{i_p},$$

where $I = \{i_1, \dots, i_p\}$ and f_I is a polynomial with coefficients in \mathbf{Q} . A global *PL form* on X is a collection of polynomial forms, one for each simplex of X , which coincide on common faces. The *PL forms* of X form a graded, commutative ($a \wedge b = (-1)^{|a||b|} b \wedge a$), differential (the differential is of degree $+1$) algebra over \mathbf{Q} (DGA), the so-called de Rham complex of X , denoted by \mathcal{E}_X .

4.2 DEFINITION OF $\underline{\mathcal{E}}_X$. Given a G -complex X , $\underline{\mathcal{E}}_X$ is a covariant functor from \mathcal{O}_G to the category of DGA's defined by $\underline{\mathcal{E}}_X(G/H) = \mathcal{E}_{X^H}$ on objects. If $\hat{a}: G/H \rightarrow G/H'$ is a morphism in \mathcal{O}_G corresponding to $a \in G$, we define $\underline{\mathcal{E}}_X(\hat{a}) = a^*: \mathcal{E}_{X^H} \rightarrow \mathcal{E}_{X^{H'}}$, where $a: X^{H'} \rightarrow X^H$ is the map given by multiplication by a .

We mentioned earlier that $\underline{C}_*(X)$ is a projective object in Vec_G . In analogy with this we have the following result concerning $\underline{\mathcal{E}}_X$ regarded as an object of Vec^* :

PROPOSITION 4.3. $\underline{\mathcal{E}}_X$ is an injective object in Vec^* .

PROOF. We shall show $\underline{\mathcal{E}}_X = \bigoplus_{H \subseteq G} V_H^*$, where $V_H^* = \bigcap_{H' \supset H} \ker \underline{\mathcal{E}}_X(\hat{e}_{H,H'})$ for just one subgroup in each conjugacy class and $V_H^* = 0$ otherwise. The set $\bigcap_{H' \supset H} \ker \underline{\mathcal{E}}_X(\hat{e}_{H,H'}) \subseteq \underline{\mathcal{E}}_{X^H}$ is the set of PL forms on X^H which are equal to 0 when restricted to $X^{H'}$ for every $H' \supset H$.

We proceed inductively on the subgroups of G . We first construct a splitting σ to $\underline{\mathcal{E}}_X \rightarrow \underline{\mathcal{E}}_{X^G}$ by considering a basis of the \mathbf{Q} -vector space $\underline{\mathcal{E}}_{X^G}$ and extending the PL forms of this basis over X . We insist that the extended PL forms vanish on each simplex of X which is disjoint from X^G . This is possible by using the extension lemmas of [2]. Let $V_G^* = \sigma(\underline{\mathcal{E}}_{X^G}) \cong \underline{\mathcal{E}}_{X^G}$. The splitting $\sigma: \underline{\mathcal{E}}_{X^G} \rightarrow \underline{\mathcal{E}}_X$ induces splittings $\sigma_H: \underline{\mathcal{E}}_{X^G} \rightarrow \underline{\mathcal{E}}_{X^H}$ as well (compose σ with the restriction map $\underline{\mathcal{E}}_X \rightarrow \underline{\mathcal{E}}_{X^H}$). Therefore

$$\underline{\mathcal{E}}_{X^H} \cong V_G^* \oplus \ker(\underline{\mathcal{E}}_{X^H} \rightarrow \underline{\mathcal{E}}_{X^G})$$

for every $H \subseteq G$. We can express this by $\underline{\mathcal{E}}_X \cong V_G^* \oplus \underline{\mathcal{E}}'_X$, where $\underline{\mathcal{E}}'_X(G/H) = \ker(\underline{\mathcal{E}}_{X^H} \rightarrow \underline{\mathcal{E}}_{X^G})$.

Now consider $H \subseteq G$ and assume that

$$\underline{\mathcal{E}}_X = \bigoplus_{H' > H} V_{H'}^* \oplus \underline{\mathcal{E}}_{X,H},$$

where $\underline{\mathcal{E}}_{X,H}(G/H) = \bigcap_{H \supset K, H' > H} \ker(\underline{\mathcal{E}}_{X^K} \rightarrow \underline{\mathcal{E}}_{X^{H'}})$; the map in the parenthesis is induced by the inclusion. Here $H' > H$ means that H is conjugate to a subgroup of H' usually written $(H') > (H)$. Let $V_H^* = \bigcap_{H' \supset H} \ker \underline{\mathcal{E}}_X(\hat{e}_{H,H'})$. Obviously $V_H^* \subseteq \underline{\mathcal{E}}_{X,H}(G/H)$. By using the extension lemmas mentioned above, we construct a splitting $V_H^* \rightarrow \underline{\mathcal{E}}_X$; we can extend the PL forms of V_H^* over X in such a way that they vanish on each simplex of X disjoint from X^H . By construction, two splittings $V_H^* \rightarrow \underline{\mathcal{E}}_X$ and $V_{H'}^* \rightarrow \underline{\mathcal{E}}_X$ are disjoint if neither $(H) \leq (H')$ nor $(H') \leq (H)$. It is easy to check that

$$\sum_{g \in G} V_H^* g^* \cong \text{Hom}_{\mathbf{Q}(N(H)/H)}(\mathbf{Q}(G/H), V_H^*)$$

and in general

$$\sum_{g \in G} V_H^* g^*|_{X^K} \cong \text{Hom}_{\mathbf{Q}(N(H)/H)}(\mathbf{Q}((G/H)^K), V_H^*);$$

g^* means the operation on PL forms induced by $g \in G$. This completes the inductive step of the proof.

DEFINITION 4.4. Let DGA_G denote the category the objects of which are covariant functors from \mathcal{O}_G to the category of simple DGA's over \mathbf{Q} , which are *injective* when regarded as objects of Vec_G^* ; the morphisms are natural transformations between these functors.

We call such functors *systems of DGA's for G*.

For instance, $\underline{\mathcal{E}}_X$ is a system of DGA's for every G -complex X . In fact, the study of $\underline{\mathcal{E}}_X$ led us to the introduction of the category DGA_G .

We can consider cohomology of a system of DGA's \mathfrak{A} in two ways.

(1) Let $\underline{H}^*(\mathfrak{A})$ be the covariant rational coefficient system defined by

$$(4.5) \quad \underline{H}^*(\mathfrak{A})(G/H) = H^*(\mathfrak{A}(G/H)).$$

(2) The cohomology of \mathfrak{A} with respect to a covariant rational coefficient system M is defined by

$$(4.6) \quad H^*(\mathfrak{A}; M) = H^*(\text{Hom}(M, \mathfrak{A}));$$

$\text{Hom}(\cdot, \cdot)$ means morphisms in Vec_G^* and is here a cochain complex (with grading and coboundary operator induced by the grading and differential of \mathfrak{A}).

As in the dual situation (2.2), there is a spectral sequence which relates the two cohomologies. We consider a projective resolution $M^{(*)}$ of M in Vec_G^* and form the double complex $\text{Hom}(M^{(*)}, \mathfrak{A})$. Standard homological algebra applied to this double complex [4] yields a spectral sequence such that

$$(4.7) \quad E_2^{s,t} = \text{Ext}^s(M, \underline{H}^t(\mathfrak{A})) \Rightarrow H^{s+t}(\mathfrak{A}; M).$$

The injectivity of \mathfrak{A} is necessary for the convergence of the spectral sequence \mathfrak{A} . This spectral sequence is a central tool for our main constructions later on.

Notice that $H^*(\mathfrak{A}; M) = \text{Hom}(M, \underline{H}^*(\mathfrak{A}))$ if M is projective. Moreover, a morphism $f: \mathfrak{A} \rightarrow \mathfrak{B}$ which induces an isomorphism $f^*: \underline{H}^*(\mathfrak{A}) \rightarrow \underline{H}^*(\mathfrak{B})$ also induces an isomorphism $f_M^*: H^*(\mathfrak{A}; M) \rightarrow H^*(\mathfrak{B}; M)$ for any $M \in \text{Vec}_G^*$ and vice versa.

Imitating the nonequivariant case, we consider relative cohomology groups

$$\underline{H}^n(\mathfrak{A}_1, \mathfrak{A}_2) = \underline{H}^n(\mathfrak{A}_1^* \times \mathfrak{A}_2^{*+1})$$

and

$$H^n(\mathfrak{A}_1, \mathfrak{A}_2; M) = H^n(\text{Hom}(M, \mathfrak{A}_1^* \times \mathfrak{A}_2^{*+1})),$$

where $\mathfrak{A}_2 \xrightarrow{i} \mathfrak{A}_1$ and $d: \mathfrak{A}_1^n \times \mathfrak{A}_2^{n+1} \rightarrow \mathfrak{A}_1^{n+1} \times \mathfrak{A}_2^{n+2}$ is equal to $(d_1 \circ \text{pr}_1 + \text{pr}_2, -d_2 \circ \text{pr}_2)$. Then there are long exact sequences

$$\cdots \rightarrow \underline{H}^n(\mathfrak{A}_2) \xrightarrow{i^*} \underline{H}^n(\mathfrak{A}_1) \xrightarrow{j^*} \underline{H}^n(\mathfrak{A}_1, \mathfrak{A}_2) \xrightarrow{\delta^*} \underline{H}^{n+1}(\mathfrak{A}_2) \rightarrow \cdots$$

and

$$(4.8) \quad \cdots \rightarrow H^n(\mathfrak{A}_2; M) \xrightarrow{i^*} H^n(\mathfrak{A}_1; M) \xrightarrow{j^*} H^n(\mathfrak{A}_1, \mathfrak{A}_2; M) \xrightarrow{\delta^*} H^{n+1}(\mathfrak{A}_2; M) \rightarrow \cdots,$$

where j^* and δ^* are induced by $j = (\text{id}, 0): \mathfrak{A}_1^n \rightarrow \mathfrak{A}_1^n \times \mathfrak{A}_2^{n+1}$ and $\text{pr}_2: \mathfrak{A}_1^n \times \mathfrak{A}_2^{n+1} \rightarrow \mathfrak{A}_2^{n+1}$ respectively.

If we take these facts into account, the proof of the equivariant de Rham theorem becomes straightforward.

THEOREM 4.9. $H_G^*(X; N) \cong H^*(\underline{\mathfrak{E}}_X; N^*)$, where X is a G -complex, N is any rational coefficient system in Vec_G and N^* is its dual in Vec_G^* .

PROOF. We denote by $\underline{C}^*(X; \mathbb{Q})$ the object in Vec_G^* defined by $\underline{C}^*(X; \mathbb{Q})(G/H) = C^*(X^H; \mathbb{Q})$, where the latter stands for the ordinary simplicial cochain complex of

X^H with coefficients in \mathbf{Q} . By integration of the PL forms of \mathcal{E}_{X^H} , $H \subseteq G$, we get a morphism $i: \underline{\mathcal{E}}_X \rightarrow \underline{C}^*(X; \mathbf{Q})$, which is compatible with the differentials. Since $i(G/H): \mathcal{E}_{X^H} \rightarrow C^*(X^H; \mathbf{Q})$ is a cohomology isomorphism (simplicial de Rham theorem) for every $H \subseteq G$, i induces an isomorphism on cohomology

$$i^*: H^*(\underline{\mathcal{E}}_X) \rightarrow H^*(\underline{C}^*(X; \mathbf{Q})) = H^*(X; \mathbf{Q}).$$

Consider the spectral sequences corresponding to the double complexes $\text{Hom}(N^{**}, \underline{\mathcal{E}}_X)$ and $\text{Hom}(N^{**}, \underline{C}^*(X; \mathbf{Q}))$. Their E_2 -terms are isomorphic:

$$\text{Ext}^s(N^*, \underline{H}^t(\underline{\mathcal{E}}_X)) \cong \text{Ext}^s(N^*, \underline{H}^t(X; \mathbf{Q})).$$

On the other hand, $\text{Ext}^s(N^*, \underline{H}^t(X; \mathbf{Q})) \cong \text{Ext}^s(\underline{H}_t(X; \mathbf{Q}), N)$ because of duality. The first spectral sequence converges to $H^*(\underline{\mathcal{E}}_X; N^*)$ and the second to $H_G^*(X; N)$. This completes the proof.

We shall finish this section by defining the notions of homotopy between morphisms in DGA_G and homotopy equivalence between systems of DGA 's.

As a motivation we remark the following: Let $f, g: X \rightarrow Y$ be two G -homotopic G -maps between G -complexes. In [1], it is proved that any G -map can be G -homotopically approximated by a simplicial G -map and any G -homotopy between two simplicial G -maps can be G -homotopically approximated by a simplicial G -homotopy. Therefore, we can assume without loss of generality that f and g are simplicial and that $H: X \times I \rightarrow Y$ is a simplicial G -homotopy between them. The maps f, g and H induce homomorphisms on the de Rham complexes $\tilde{f}, \tilde{g}: \mathcal{E}_Y \rightarrow \mathcal{E}_X$ and $\tilde{H}: \mathcal{E}_Y \rightarrow \mathcal{E}_{X \times I} \cong \mathcal{E}_X \otimes \mathbf{Q}(t, dt)$; $\mathbf{Q}(t, dt)$ is the free DGA on two generators t and dt of degree 0 and 1 respectively with $d(t) = dt$. Moreover, $H_0 = f$ and $H_1 = g$ imply $e_0 \circ \tilde{H} = \tilde{f}$ and $e_1 \circ \tilde{H} = \tilde{g}$, where e_0 is the projection $\mathcal{E}_X \otimes \mathbf{Q}(t, dt) \rightarrow \mathcal{E}_X$ with $t = 0, dt = 0$ and e_1 the projection with $t = 1, dt = 0$. The restrictions of f, g and H on the fixed point sets X^K and $X^K \times I$ induce homomorphisms $\mathcal{E}_{Y^K} \rightarrow \mathcal{E}_{X^K}$ and $\mathcal{E}_{Y^K} \rightarrow \mathcal{E}_{X^K} \otimes \mathbf{Q}(t, dt)$ as well, which have the above property. Furthermore, this procedure is functorial.

Given a system of DGA 's \mathfrak{A} , let $\mathfrak{A}(t, dt)$ be the system of DGA 's defined by $\mathfrak{A}(t, dt)(G/H) = \mathfrak{A}(G/H)(t, dt)$.

DEFINITION 4.10. Two morphisms $f, g: \mathfrak{A} \rightarrow \mathfrak{B}$ are homotopic if there is a morphism $H: \mathfrak{A} \rightarrow \mathfrak{B}(t, dt)$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$ where e_0 is the projection $t = 0, dt = 0$ and e_1 the projection $t = 1, dt = 0$.

COROLLARY 4.11. Any two simplicial G -homotopic G -maps $f, g: X \rightarrow Y$ induce homotopic morphisms $\tilde{f}, \tilde{g}: \mathcal{E}_Y \rightarrow \mathcal{E}_X$.

PROOF. It follows directly from our discussion above.

As in the ordinary case, the homotopy relation defined in 4.10 is not an equivalence relation in the set of morphisms from \mathfrak{A} to \mathfrak{B} . Instead, we have

DEFINITION 4.12. Two systems of DGA 's \mathfrak{A} and \mathfrak{B} are *weakly equivalent* if there is a chain of cohomology isomorphisms

$$\mathfrak{A} \xrightarrow{f_1} \mathfrak{C}_1 \xrightarrow{f_2} \mathfrak{C}_2 \leftarrow \cdots \rightarrow \mathfrak{B}$$

connecting them.

By means of the minimal systems of DGA's, which will be introduced in the next section, we shall prove that weak equivalence classes of systems of DGA's for G correspond bijectively to rational homotopy types of G -spaces.

5. Minimal systems of DGA's. Recall that a (connected) DGA \mathfrak{M} is minimal if it is free and its differential is decomposable. Freeness means that $\mathfrak{M} = \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n \otimes \cdots$, where

$$\mathfrak{M}_n = \begin{cases} \text{a polynomial algebra generated by elements of degree } p, & p \text{ even,} \\ \text{an exterior algebra generated by elements of degree } p, & p \text{ odd} \end{cases}$$

and decomposability of the differential d means that $d(\mathfrak{M}) \subseteq \mathfrak{M}^+ \wedge \mathfrak{M}^+$, where \mathfrak{M}^+ is the set of positive degree elements of \mathfrak{M} . Let $\mathfrak{M}(n) = \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$. \mathfrak{M} is called simple if d restricted to $\mathfrak{M}(n)$ takes values in $\mathfrak{M}(n-1)$ for every n . We restrict attention to simple DGA's, i.e., to DGA's the minimal model of which is simple.

DEFINITION 5.1. A system of DGA's \mathfrak{M} is minimal if it satisfies the following properties:

- (i) $\mathfrak{M}(G/H)$ is a free DGA for all $H \subseteq G$,
- (ii) $\mathfrak{M}(G/G)$ is a minimal DGA (initial condition), and
- (iii) $d(\cap_{H' \supset H} \ker \mathfrak{M}(\hat{e}_{H,H'})) \subseteq \mathfrak{M}(G/H)$ is decomposable, where $\hat{e}_{H,H'}: G/H \rightarrow G/H'$ is the projection.

We shall show that the minimal systems of DGA's for G play the same role in the category of DGA_G as the minimal DGA's in the category of DGA's. In this section, we shall state the results and in §7, we shall prove them.

THEOREM 5.2. *If $f: \mathfrak{M} \rightarrow \mathfrak{N}$ is a cohomology isomorphism between systems of DGA's, then f is an isomorphism.*

THEOREM 5.3. *For any system of DGA's \mathfrak{A} there exist a minimal system of DGA's \mathfrak{M} and a morphism $\rho: \mathfrak{M} \rightarrow \mathfrak{A}$ inducing an isomorphism on cohomology.*

The following propositions describe the properties of minimal systems of DGA's with respect to homotopy. They are completely analogous to the nonequivariant ones.

PROPOSITION 5.4. *The homotopy relation in the set of morphisms from a minimal system of DGA's to an arbitrary system of DGA's is an equivalence relation.*

PROPOSITION 5.5. *Let $\pi: \mathfrak{A} \rightarrow \mathfrak{B}$ be a cohomology isomorphism and let $f: \mathfrak{M} \rightarrow \mathfrak{B}$ be an arbitrary morphism, where \mathfrak{M} is minimal. Then there is a morphism $f': \mathfrak{M} \rightarrow \mathfrak{A}$, unique up to homotopy, such that the diagram*

$$\begin{array}{ccc} & & \mathfrak{A} \\ & \nearrow f' & \downarrow \pi \\ \mathfrak{M} & \xrightarrow{f} & \mathfrak{B} \end{array}$$

commutes up to homotopy.

COROLLARY 5.6. *If $\pi: \mathfrak{A} \rightarrow \mathfrak{B}$ is a cohomology isomorphism, then it induces a bijection*

$$\pi_*: [\mathfrak{M}, \mathfrak{A}] \rightarrow [\mathfrak{M}, \mathfrak{B}],$$

where $[\mathfrak{M}, \mathfrak{A}]$ is the set of homotopy classes of morphisms from \mathfrak{M} to \mathfrak{A} .

COROLLARY 5.7. *Let $\rho: \mathfrak{M} \rightarrow \mathfrak{A}$ and $\rho': \mathfrak{M}' \rightarrow \mathfrak{A}$ be two cohomology isomorphisms, where \mathfrak{M} and \mathfrak{M}' are minimal systems of DGA's. Then there is an isomorphism $i: \mathfrak{M} \rightarrow \mathfrak{M}'$ such that the diagram*

$$\begin{array}{ccc} & \mathfrak{M}' & \\ \uparrow i & \searrow \rho' & \\ \mathfrak{M} & \xrightarrow{\rho} & \mathfrak{A} \end{array}$$

commutes up to homotopy.

This uniqueness result makes the following definition meaningful.

DEFINITION 5.8. Let $\rho: \mathfrak{M} \rightarrow \mathfrak{A}$ be a cohomology isomorphism, where \mathfrak{M} is minimal. Then \mathfrak{M} is called the *minimal model* of \mathfrak{A} .

COROLLARY 5.9. *In each weak equivalence class of systems of DGA's for G there is one and, up to isomorphism, only one minimal system of DGA's.*

The following construction is central to the analysis of minimal systems of DGA's and mirrors the topological construction of principal G -fibrations (see Lemmas 6.3 and 6.4 below). It is an important tool in the proof of Theorem 5.3.

CONSTRUCTION 5.10. Let \mathfrak{A} be a system of DGA's for G , let W_n be an object in Vec_G^* and let $d: W_n \rightarrow \underline{Z}^{n+1}(\mathfrak{A})$ be a morphism in Vec_G^* , where $\underline{Z}^{n+1}(\mathfrak{A})(G/H)$ is the set of cocycles of degree $n+1$ of the DGA $\mathfrak{A}(G/H)$. We construct a new system of DGA's $\mathfrak{A}(W_n)$, called the *elementary extension* of \mathfrak{A} with respect to d , as follows. Let

$$0 \rightarrow W_n \xrightarrow{w} W_{n,0} \xrightarrow{w_0} W_{n,1} \rightarrow \cdots W_{n,i} \xrightarrow{w_i} \cdots$$

be a minimal injective resolution of W_n in Vec_G^* , i.e., an injective resolution such that $W_{n,i}$ is the injective envelope of $\text{im } w_{i-1}$. One can see immediately that d induces morphisms $d_i: W_{n,i} \rightarrow \mathfrak{A}^{n+i+1}$ such that the diagram

$$\begin{array}{ccccccc} 0 \rightarrow & W_n & \xrightarrow{w} & W_{n,0} & \xrightarrow{w_0} & W_{n,1} & \rightarrow \cdots W_{n,i} \xrightarrow{w_i} \cdots \\ & d \downarrow & & d_0 \downarrow & & d_1 \downarrow & & d_i \downarrow \\ & \underline{Z}^{n+1}(\mathfrak{A}) & \rightarrow & \mathfrak{A}^{n+1} & \rightarrow & \mathfrak{A}^{n+2} & \rightarrow \cdots \mathfrak{A}^{n+i+1} \rightarrow \cdots \end{array}$$

commutes. Let $W_{n,*} = \bigoplus_{m \geq 0} W_{n,m}$, where $W_{n,i}$ is considered as being concentrated in degree $n+i$. Define an object $\Lambda(W_{n,*})$ in Vec_G^* by

$$\Lambda(W_{n,*})(G/H) = \Lambda(W_{n,*}(G/H)),$$

where the latter is the free graded commutative algebra generated by the graded vector space $W_{n,*}(G/H)$. We define $\mathfrak{A}(W_n) = \mathfrak{A} \otimes \Lambda(W_{n,*})$, where the differential

of $\mathfrak{A}(W_n)$ restricts on \mathfrak{A} to the given differential and restricts on $W_{n,i}$ to $w_i + (-1)^i d_i$, $i \geq 0$. The object $\mathfrak{A}(W_n)$ is injective as it is the tensor product of injectives.

REMARK. The injective resolution of W_n corresponds by duality to the projective resolution in the construction of Eilenberg-Mac Lane G -complexes.

As an immediate corollary of the proof of Theorem 5.3 we shall have the following description of minimal systems of DGA's:

COROLLARY 5.11. *Any minimal system of DGA's \mathfrak{M} can be written as a union of an expanding sequence*

$$\mathfrak{M}(1) \subseteq \cdots \subseteq \mathfrak{M}(n-1) \subseteq \mathfrak{M}(n) \subseteq \cdots$$

of minimal systems of DGA's, where $\mathfrak{M}(n)$ is an elementary extension $\mathfrak{M}(n) = \mathfrak{M}(n-1)(W_n)$. Moreover,

$$W_n \cong \underline{H}^n(\mathfrak{M}(n), \mathfrak{M}(n-1)).$$

6. Equivariant minimal models: the main theorems. In this section we establish the relation between G -spaces and minimal systems of DGA's for G .

DEFINITION 6.1. The *equivariant minimal model* \mathfrak{M}_X of a G -complex X is the minimal model of the system of DGA's $\underline{\mathfrak{G}}_X$.

The main results of this paper are the following Theorems 6.2 and 6.5:

THEOREM 6.2. *The correspondence $X \mapsto \mathfrak{M}_X$ induces a bijection between equivariant rational homotopy types of G -spaces on the one hand and isomorphism classes of minimal systems of DGA's for G on the other. Moreover, the following relations hold.*

(1) $H_G^*(X; N) \cong H^*(\underline{\mathfrak{G}}_X; N^*)$, where N is a coefficient system in Vec_G and N^* is its dual in Vec_G^* .

(2) $\pi_n(X) \otimes \mathbb{Q} \cong W_n^*$, where $\mathfrak{M}_X(n) = \mathfrak{M}_X(n-1)(W_n)$ and W_n^* is the dual of W_n in Vec_G^* .

(3) If

$$\begin{array}{c} X \\ \swarrow \quad \searrow \quad \searrow f_n \\ \{pt\} \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots \end{array}$$

is an equivariant Postnikov decomposition of X then

- (i) $\mathfrak{M}_X(n)$ is the equivariant minimal model of X_n and
- (ii) the morphism

$$W_n \xrightarrow{d} \underline{Z}^{n+1}(\mathfrak{M}_X(n-1))$$

determines the equivariant k -invariant

$$k^{n+1} \in H_G^{n+1}(X_{n-1}; \pi_n(X)) \otimes \mathbb{Q} \cong H_G^{n+1}(X_{0,n-1}; \pi_n(X_0))$$

of the equivariant rationalization X_0 of X .

PROOF. Relation (1) is an immediate consequence of the equivariant de Rham theorem.

The proof of the bijection stated in the theorem is based on the following two lemmas. They establish the equivalence between elementary extensions of systems of DGA's and principal G -fibrations. We shall state them here and prove them in §8.

Say that a system of DGA's \mathfrak{A} is *geometric* for X if there is a cohomology isomorphism $\mathfrak{A} \rightarrow \underline{\mathbb{G}}_X$.

LEMMA 6.3 (analogue to the Hirsch lemma). *Let $F = K(M, n) \rightarrow E \xrightarrow{p} X$ be a G -fibration and let \mathfrak{A} be a system of DGA's geometric for X . Then there is an elementary extension $\tilde{\mathfrak{A}} = \mathfrak{A}(W_n)$ of \mathfrak{A} which is geometric for E . Moreover, $W_n = H^n(F; \mathbb{Q})$.*

LEMMA 6.4 (converse to the previous one). *Let X be a G -complex and let \mathfrak{A} be a system of DGA's geometric for X . If $\tilde{\mathfrak{A}}$ is an elementary extension of \mathfrak{A} with respect to $\tilde{d}: W_n \rightarrow \mathbb{Z}^{n+1}(\mathfrak{A})$ there is a G -fibration $K(W_n^*, n) \rightarrow E \rightarrow X$ such that $\tilde{\mathfrak{A}}$ is geometric for E .*

Let \mathfrak{M} be a minimal system of DGA's for G . Then there is an expanding sequence of elementary extensions

$$\mathbb{Q} \subseteq \mathfrak{M}(1) \subseteq \cdots \subseteq \mathfrak{M}(n-1) \subseteq \mathfrak{M}(n) \subseteq \cdots$$

such that $\mathfrak{M} = \bigcup_{n \geq 0} \mathfrak{M}(n)$ (Corollary 5.11). According to the previous lemma, we can construct inductively (starting from a point) a sequence of principal G -fibrations, i.e., an equivariant Postnikov decomposition

$$\{pt\} \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots$$

The inverse limit of it is a G -space. The barycentric subdivision of the semisimplicial resolution of this space is a G -complex.

Conversely, let X be a G -complex. According to the first lemma, the G -Postnikov decomposition of X induces an expanding sequence of elementary extensions of minimal systems of DGA's for G (starting from \mathbb{Q} , $\mathbb{Q}(G/H) = \mathbb{Q}$). Moreover, the properties (2) and (3) of the theorem are valid by construction. The uniqueness arguments are analogous to the nonequivariant ones and are based on 2.5 and 5.7.

THEOREM 6.5. *There exists a bijection $[X, Y]_G \cong [\mathfrak{M}_Y, \mathfrak{M}_X]$, where X is a G -space and Y is a \mathbb{Q} - G -space.*

The proof consists of the following two propositions.

PROPOSITION 6.6. *Let $f: X \rightarrow Y$ be a G -map between two G -complexes and let $\rho_X: \mathfrak{M}_X \rightarrow \underline{\mathbb{G}}_X$ and $\rho_Y: \mathfrak{M}_Y \rightarrow \underline{\mathbb{G}}_Y$ be equivariant minimal models of X and Y respectively. Then there exists a morphism $\tilde{f}: \mathfrak{M}_Y \rightarrow \mathfrak{M}_X$ such that the diagram*

$$\begin{array}{ccc} \underline{\mathbb{G}}_Y & \xrightarrow{f^*} & \underline{\mathbb{G}}_X \\ \rho_Y \uparrow & & \uparrow \rho_X \\ \mathfrak{M}_Y & \xrightarrow{\tilde{f}} & \mathfrak{M}_X \end{array}$$

commutes up to homotopy. Moreover, \tilde{f} is unique up to homotopy and depends only on the rational G -homotopy class of f .

PROOF. This is an immediate consequence of Corollary 4.11 and Proposition 5.5. Just as in the nonequivariant case, the last statement follows from a relative version of Proposition 5.5.

PROPOSITION 6.7. *Let $\tilde{f}: \mathfrak{N} \rightarrow \mathfrak{M}$ be a morphism between two minimal systems of DGA's for G , where \mathfrak{M} and \mathfrak{N} are the equivariant minimal models of the rational G -spaces X and Y respectively. Then there is a G -map $f: X \rightarrow Y$, such that the diagram*

$$\begin{array}{ccc} \mathbb{G}_Y & \xrightarrow{\tilde{f}^*} & \mathbb{G}_X \\ \rho_Y \uparrow & & \uparrow \rho_X \\ \mathfrak{N} & \xrightarrow{\tilde{f}} & \mathfrak{M} \end{array}$$

commutes up to homotopy. Moreover, f is unique up to homotopy and depends only on the homotopy class of \tilde{f} .

The proof is deferred to §8.

7. Deferred proofs from §5.

PROOF OF PROPOSITION 5.2. Let $f: \mathfrak{M} \rightarrow \mathfrak{N}$ be a cohomology isomorphism. Since $\mathfrak{M}(G/G)$ and $\mathfrak{N}(G/G)$ are minimal DGA's and $f(G/G)$ is a cohomology isomorphism between them, $f(G/G)$ is an isomorphism. Given $H \subseteq G$, assume inductively that $f(G/H'): \mathfrak{M}(G/H') \rightarrow \mathfrak{N}(G/H')$ is an isomorphism for all subgroups $H' \supset H$. We shall prove that $f(G/H)$ is also an isomorphism. Let $f(G/H)$ be an isomorphism when restricted to $\bigcup_{k \leq n-1} \mathfrak{M}(G/H)^k$.

Injectivity of $f(G/H)^n$: $\mathfrak{M}(G/H)^n \rightarrow \mathfrak{N}(G/H)^n$. Let $x \in \mathfrak{M}(G/H)^n$ be such that $f(G/H)(x) = 0$. From the injectivity of $f(G/H')$, $H' \supset H$, and the relation $f(G/H') \circ \mathfrak{M}(\hat{e}_{H,H'})(x) = \mathfrak{N}(\hat{e}_{H,H'}) \circ f(G/H)(x) = 0$ it follows that $\mathfrak{M}(\hat{e}_{H,H'})(x) = 0$ for every $H' \supset H$. Therefore $dx \in \mathfrak{M}(G/H)^+ \cdot \mathfrak{M}(G/H)^+$, i.e., dx can be expressed as a sum of products of elements of degree less than n . Since $f(G/H)$ is injective in lower degrees and the algebras are free, we have $dx = 0$. Because of the injectivity of $f(G/H)^*$ on the cohomology we have $x = dt$, $t \in \mathfrak{M}(G/H)^{n-1}$. Then $df(G/H)(t) = f(G/H)(dt) = 0$, so $f(G/H)(t)$ is the cocycle. The surjectivity of $f(G/H)^*$ on cohomology implies that there is a cocycle $t' \in \mathfrak{M}(G/H)^{n-1}$ such that $f(G/H)(t) - f(G/H)(t') = du$, $u \in \mathfrak{N}(G/H)^{n-2}$. Therefore we can assume that

$$f(G/H)(t) = du, \quad u \in \mathfrak{N}(G/H)^{n-2}.$$

Since $f(G/H)$ is an isomorphism in degree $n-2$, there is $u' \in \mathfrak{M}(G/H)^{n-2}$ such that $f(G/H)(u') = u$. Then du' is a preimage of $f(G/H)(t)$ and, because of the injectivity of $f(G/H)$ in degree $n-1$, it must be equal to t . Then $dt = ddu' = 0$, i.e., $x = 0$.

Surjectivity of $f(G/H)^n$. Let $y \in \mathfrak{N}(G/H)^n$. We shall show that $y = f(G/H)(x)$, $x \in \mathfrak{M}(G/H)^n$. Let $y_{H'} = \mathfrak{N}(\hat{e}_{H,H'})(y) \in \mathfrak{N}(G/H')$, $H' \supset H$, and consider the preimages $\bar{y}_{H'} \in \mathfrak{M}(G/H)^n$.

Claim. There is an element $\bar{y} \in \mathfrak{M}(G/H)^n$ such that

$$\mathfrak{M}(\hat{e}_{H,H'})(\bar{y}) = \bar{y}_{H'} \quad \text{for all } H' \supset H.$$

Granting the claim, replace y by $y - f(G/H)(\bar{y})$ and so assume that $\mathfrak{N}(\hat{e}_{H,H'})(y) = 0$ for every $H' \supset H$. Then dy is decomposable and it has a preimage z , $f(G/H)(z) = dy$, because $f(G/H)$ is surjective in lower degrees. Moreover, z and dz are decomposable, n being the highest degree of indecomposable elements appearing in the expression of dz . Since $f(G/H)(dz) = 0$ and $f(G/H)$ is injective in degrees less than or equal to n , $dz = 0$. Moreover, z is a coboundary because $f^*(G/H)$ is injective. Let $z = dv$, $v \in \mathfrak{M}(G/H)^n$. If we replace y by $y - f(G/H)(v)$ we can assume $dy = 0$. Then there is a $y' \in \mathfrak{M}(G/H)^n$ such that $f(G/H)(y') - y = du$. So we can assume

$$y = du, \quad u \in \mathfrak{N}(G/H)^{n-1}.$$

But u has a preimage $u' \in \mathfrak{M}(G/H)^{n-1}$. Then du' is a preimage of y , i.e., $f(G/H)(du') = y$.

PROOF OF THE CLAIM. \mathfrak{M}^n is injective by assumption. Therefore we can write $\mathfrak{M}^n = \bigoplus_{K \subseteq G} V_K^*$. In particular,

$$\mathfrak{M}^n(G/H) = \bigoplus_{K \subseteq G} V_K^*(G/H) = V_H^* \oplus \bigoplus_{H' > H} \text{Hom}_{\mathbf{Q}(N(H')/H')}(\mathbf{Q}((G/H')^H), V_{H'}^*),$$

because $(G/K)^H = 0$ if H is not conjugate to a subgroup of K . Since $f(G/H')$ is an isomorphism for all $H' > H$ and $f(G/H)^n: \mathfrak{M}(G/H)^n \rightarrow \mathfrak{N}(G/H)^n$ is injective we can write

$$\mathfrak{N}^n(G/H) = V_H^* \oplus \bigoplus_{H' > H} \text{Hom}_{\mathbf{Q}(N(H')/H')}(\mathbf{Q}((G/H')^H), V_{H'}^*) \oplus W_H^*.$$

Moreover, we can choose W_H^* such that $W_H^* \subseteq \bigcap_{H' \supset H} \ker \mathfrak{N}(\hat{e}_{H,H'})$ using the fact that \mathfrak{N}^n is injective. Consider the projection of y on

$$V_H^* \oplus \bigoplus_{H' > H} \text{Hom}_{\mathbf{Q}(N(H')/H')}(\mathbf{Q}((G/H')^H), V_{H'}^*).$$

It has a preimage \bar{y} in $\mathfrak{M}^n(G/H)$. Then $\mathfrak{N}(\hat{e}_{H,H'})(\bar{y}) = \bar{y}_{H'}$ for all $H' \supset H$.

PROOF OF THEOREM 5.3. (Existence of the minimal model): We proceed inductively on degree. Suppose there is a minimal system of DGA's $\mathfrak{M}(n-1)$ and a morphism $\rho_{n-1}: \mathfrak{M}(n-1) \rightarrow \mathfrak{A}$ such that $\rho_{n-1}^k: \underline{H}^k(\mathfrak{M}(n-1)) \rightarrow \underline{H}^k(\mathfrak{A})$ is an isomorphism if $k \leq n-1$ and a monomorphism if $k = n$.

We consider $W'_n = \text{coker } \rho_{n-1}^n$ in Vec_G^* , where

$$\text{coker } \rho_{n-1}^n(G/H) = \text{coker } \rho_{n-1}^n(G/H).$$

Let $p: P \rightarrow W'_n$ be an epimorphism, where P is projective. There exist liftings p' and p'' in the following diagram by projectivity; here $\underline{Z}^n(\mathfrak{A})$ is the functor in Vec_G^* defined by $\underline{Z}^n(\mathfrak{A})(G/H) = Z^n(\mathfrak{A}(G/H))$, where $Z^n()$ means the set of cocycles of degree n .

$$\begin{array}{ccccccc}
 & & \ker p & \xrightarrow{i} & P & & \\
 & \swarrow p' \circ i & & & \downarrow p & & \\
 0 & \longrightarrow & \underline{H}^n(\mathfrak{M}(n-1)) & \longrightarrow & \underline{H}^n(\mathfrak{A}) & \longrightarrow & \text{coker } \rho_{n-1}^n = W'_n \longrightarrow 0 \\
 & \uparrow q & & & \uparrow p'' & & \\
 & \underline{Z}^n(\mathfrak{M}(n-1)) & \xrightarrow{\rho_{n-1}} & \underline{Z}^n(\mathfrak{A}) & & &
 \end{array}$$

The existence of the morphism q , such that the entire diagram commutes, follows from the existence of p' and p'' by comparing the two spectral sequences

$$E_2^{s,t} = \text{Ext}^s(\underline{H}^t(\mathfrak{M}(n-1)), \underline{\ker p})^{\rho_{n-1}^{s-1}} \rightarrow \text{Ext}^s(\underline{H}^t(\mathfrak{A}), \underline{\ker p}) = E_2'^{s,t}.$$

Because of our assumption on ρ_{n-1} , we have $E_2^{s,t} \cong E_2'^{s,t}$ for every s and for $t \leq n-1$. Moreover, $E_2^{0,n} \subseteq E_2'^{0,n}$ and $d_r'^{0,n}|_{E_2^{0,n}} = d_r^{0,n}$ for every r . Thus

$$(7.1) \quad E_\infty^{s,t} \cong E_\infty'^{s,t}$$

for $s+t \leq n$, $s \neq 0$ and $E_\infty^{0,n} \subseteq E_\infty'^{0,n}$. In particular, we consider the edge homomorphisms

$$\begin{array}{ccccccc} H^n & \twoheadrightarrow & H^n/F'H^n & = & E_\infty^{0,n} \subseteq & \cdots & \subseteq E_3^{0,n} \subseteq E_2^{0,n} \\ \downarrow & & & & \cap & & \cap \\ H'^n & \twoheadrightarrow & H'^n/F'H'^n & = & E_\infty'^{0,n} \subseteq & \cdots & \subseteq E_3'^{0,n} \subseteq E_2'^{0,n} \end{array}$$

where H^n and H'^n are equal to $H^n(\mathfrak{M}(n-1), \underline{\ker p})$ and $H^n(\mathfrak{A}, \underline{\ker p})$ respectively. The isomorphism (7.1) implies $F^i H^n = F^i H'^n$ for every $i \geq 1$ by induction; therefore, $H^n \subseteq H'^n$.

Consider $[p' \circ i] \in E_2^{0,n} \subseteq E_2'^{0,n}$. Since $[p' \circ i]$ has a preimage $[p'' \circ i] \in H'^n$, it belongs to $E_\infty'^{0,n}$. This means that $d_r'([p' \circ i]) = 0$ for every r . But then also $d_r([p' \circ i]) = 0$ for every r . Therefore $[p' \circ i]$ has a preimage $[q] \in H^n$. The difference between $p'' \circ i$ and $\rho_{n-1} \circ q$ by a coboundary can be easily corrected.

Since $\mathfrak{M}(n-1)^n$ is injective in Vec_G^* , the morphism $q: \underline{\ker p} \rightarrow \underline{Z}^n(\mathfrak{M}(n-1)) \subseteq \mathfrak{M}(n-1)^n$ can be extended to a morphism $q': P \rightarrow \mathfrak{M}(n-1)^n$. The morphisms $d \circ q': P \rightarrow \mathfrak{M}(n-1)^{n+1}$ and $\rho_{n-1} \circ q' - p'': P \rightarrow \mathfrak{A}^n$ are equal to 0 when restricted to $\underline{\ker p}$. Therefore they factor through W'_n . We call the new morphisms $d': W'_n \rightarrow \mathfrak{M}(n-1)^{n+1}$ and $\rho'_n: W'_n \rightarrow \mathfrak{A}^n$ respectively.

As in construction 5.10, we consider a minimal injective resolution of W'_n in Vec_G^*

$$0 \rightarrow W'_n \xrightarrow{w'_0} W'_{n,0} \xrightarrow{w'_1} W'_{n,1} \rightarrow \cdots \rightarrow W'_{n,i} \xrightarrow{w'_i} \cdots$$

and construct an elementary extension $\mathfrak{M}(n-1)(W'_n)$ of $\mathfrak{M}(n-1)$ with respect to d' : $W'_n \rightarrow \mathfrak{M}(n-1)^{n+1}$. Then ρ'_n induces morphisms $\rho'_{n,i}: W'_{n,i} \rightarrow \mathfrak{A}^{n+1}$ compatible with the differential of $\mathfrak{M}(n-1)(W'_n)$. Therefore we can extend ρ_{n-1} and ρ'_n to a morphism between systems of DGA's which we call again $\rho'_n: \mathfrak{M}(n-1)(W'_n) \rightarrow \mathfrak{A}$. It is easy to check that this morphism induces an isomorphism on \underline{H}^n . In order to do so, we consider the effect of the above construction on each G/H , $H \subseteq G$, and so reduce the verification to the nonequivariant case.

Now let W''_n be the kernel of $\rho'^{n+1}_n: H^{n+1}(\mathfrak{M}(n-1)(W'_n)) \rightarrow H^{n+1}(\mathfrak{A})$ and let $q: Q \rightarrow W''_n$ be an epimorphism, where Q is a projective object in Vec_G^* . In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W'_n = \underline{\ker \rho'^{n+1}_n} & \xrightarrow{i} & \underline{H}^{n+1}(\mathfrak{M}(n-1)(W'_n)) & \longrightarrow & \underline{H}^{n+1}(\mathfrak{A}) \\ & & \uparrow q & & \uparrow & & \uparrow \\ & & 0 & \xrightarrow{q'} & \underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n)) & \longrightarrow & \underline{Z}^{n+1}(\mathfrak{A}) \\ & & \uparrow i & & \uparrow d & & \uparrow r \\ & & \underline{\ker q} & \xrightarrow{q''} & \mathfrak{M}(n-1)(W'_n)^n & \xrightarrow{\rho'_n} & \mathfrak{A}^n \end{array}$$

q' exists because \mathbf{Q} is projective. We claim that the morphism $q' \circ i: \ker q \rightarrow \underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n))$ is a coboundary with respect to the cohomology $\overline{H}^{n+1}(\mathfrak{M}(n-1)(w'_n); \ker q)$, i.e., it factors through the differential d via some q'' : $\ker q \rightarrow \mathfrak{M}(n-1)(W'_n)^n$. Indeed, the morphism $\rho'_n \circ q'$ can be regarded as $\rho'_n \circ q': \underline{Q} \rightarrow \underline{B}^{n+1}(\mathfrak{A})$, where $\underline{B}^{n+1}(\mathfrak{A})(G/H)$ is the set of coboundaries of degree $n+1$ of the DGA $\mathfrak{A}(G/H)$. Since $d: \mathfrak{A}^n \rightarrow \underline{B}^{n+1}(\mathfrak{A})$ is an endomorphism, there exists $r: Q \rightarrow \mathfrak{A}^n$ such that $\rho'_n \circ q' = d \circ r$. This implies that $\rho'_n \circ q' \circ i$ is a coboundary with respect to the cohomology $H^{n+1}(\mathfrak{A}; \ker q)$. The cohomology class $[q' \circ i]$ lies in the kernel of

$$\rho_n^{n+1}: H^{n+1}(\mathfrak{M}(n-1)(W'_n); \ker q) \rightarrow H^{n+1}(\mathfrak{A}; \ker q)$$

and in the kernel of the edge of homomorphism

$$H^{n+1}(\mathfrak{M}(n-1)(W'_n); \ker q) \rightarrow \text{Hom}(\ker q, H^{n+1}(\mathfrak{M}(n-1)(W'_n))).$$

We again study the relation between the corresponding spectral sequences and we find that the only common element of the two kernels is 0. This proves the claim above and the existence of q'' .

We extend q'' to a morphism $\hat{q}: Q \rightarrow \mathfrak{M}(n-1)(W'_n)^n$. The morphisms $q' - d \circ \hat{q}$ and $r - \rho'_n \circ \hat{q}$ are 0 when restricted to $\ker q$. Therefore, they factor through q via $d'': W''_n \rightarrow \underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n))$ and $\rho''_n: W''_n \rightarrow \mathfrak{A}^n$ respectively. Again, we consider the elementary extension of $\mathfrak{M}(n-1)(W'_n)$ with respect to d''

$$\mathfrak{M}(n) \equiv \mathfrak{M}(n-1)(W'_n)(W''_n)$$

and extend $\rho'_n: \mathfrak{M}(n-1)(W'_n) \rightarrow \mathfrak{A}$ and ρ''_n to a morphism $\rho_n: \mathfrak{M}(n) \rightarrow \mathfrak{A}$.

The *minimality* of $\mathfrak{M}(n)$ can be seen as follows. We want to prove that the differential of $\mathfrak{M}(n)$, when restricted on $\cap_{H' \supset H} \ker \mathfrak{M}(n)(\hat{e}_{H,H'}) \subseteq \mathfrak{M}(n)(G/H)$, is decomposable for every $H \subseteq G$. Since d restricted on $\cap_{H' \supset H} \ker \mathfrak{M}(n-1)(\hat{e}_{H,H'})$ is decomposable by assumption, we have to check the differential on $\cap_{H' \supset H} \ker W'_{n,i}(\hat{e}_{H,H'})$ and on $\cap_{H' \supset H} \ker W''_{n,i}(\hat{e}_{H,H'})$ for every $i \geq 0$. We start with $i = 0$. We first note that if M is in Vec_G^* and I is its injective envelope then

$$\bigcap_{H' \supset H} \ker M(\hat{e}_{H,H'}) \cong \bigcap_{H' \supset H} \ker I(\hat{e}_{H,H'})$$

for all $H \subseteq G$ by our construction of injective envelopes. In our case

$$(7.2) \quad \bigcap_{H' \supset H} \ker W'_n(\hat{e}_{H,H'}) \cong \bigcap_{H' \supset H} \ker W'_{n,0}(\hat{e}_{H,H'});$$

the same is true for W''_n . The differential d' on W'_n is defined as $d \circ q'$, where d is the differential of $\mathfrak{M}(n-1)$. Therefore it is decomposable when restricted to (7.2). The differential d'' on W''_n is defined as $q' - d \circ \hat{q}$, where d is the differential of $\mathfrak{M}(n-1)$. We only need to prove that $q'(\cap_{H' \supset H} \ker W''_{n,i}(\hat{e}_{H,H'}))$ is decomposable. In fact, the cocycle $q': Q \rightarrow \underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n))$ can be chosen to take values in the decomposable part of $\underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n))$.

In order to explain this, we first recall some facts about free DGA's. If \mathcal{A} is a free DGA, it can be written $\mathcal{A} = \mathfrak{M} \otimes \mathbf{Q}(V, dV)$, where \mathfrak{M} is a minimal DGA and

$\mathbf{Q}(V, dV)$ is a free acyclic DGA generated by the graded vector space $V \oplus dV$ as an algebra and such that $d(V) = dV$. If $\mathfrak{M} = \mathfrak{M}(n)$, i.e., \mathfrak{M} is generated by elements of degree less than or equal to n , then $Z^{n+1}(\mathcal{Q}) \cap \mathcal{Q}^+ \wedge \mathcal{Q}^+ \rightarrow H^{n+1}(\mathcal{Q})$ is surjective (every cohomology class can be represented by a decomposable cocycle). This is the case for each $\mathfrak{M}(n-1)(W'_n)(G/H)$, $H \subseteq G$, i.e.,

$$\begin{aligned} \underline{Z}^{n+1}(\mathfrak{M}(n-1)(W'_n)) \cap \mathfrak{M}(n-1)(W'_n)^+ \wedge \mathfrak{M}(n-1)(W'_n)^+ \\ \rightarrow \underline{H}^{n+1}(\mathfrak{M}(n-1)(W'_n)) \end{aligned}$$

is an epimorphism. Therefore $i \circ q: Q \rightarrow \underline{H}^{n+1}(\mathfrak{M}(n-1)(W'_n))$ has a lifting with respect to this epimorphism. Call the lifting q' .

For $i > 0$, similar arguments apply. This completes the proof of the minimality of $\mathfrak{M}(n)$.

In order to prove that

$$(7.3) \quad \rho_n^n: \underline{H}^n(\mathfrak{M}(n)) \rightarrow \underline{H}^n(\mathfrak{A})$$

is an isomorphism and that

$$(7.4) \quad \rho_n^{n+1}: \underline{H}^{n+1}(\mathfrak{M}(n)) \rightarrow H^{n+1}(\mathfrak{A})$$

is a monomorphism, we study the above construction for each G/H , $H \subseteq G$, separately. This presents no difficulty and we will omit the proof.

PROOF OF COROLLARY 5.11. Let \mathfrak{M} be a minimal system of DGA's for G . According to the constructions of the proof of Theorem 5.3 there is a sequence

$$\mathfrak{M}(1) \subseteq \cdots \subseteq \mathfrak{M}(n-1) \subseteq \mathfrak{M}(n) \subseteq \cdots$$

of minimal systems of DGA's and morphisms $\rho_n: \mathfrak{M}(n) \rightarrow \mathfrak{A}$ with the properties (7.3) and (7.4). Consider the union $\mathfrak{M}' = \bigcup_{n \geq 0} \mathfrak{M}(n)$ and the induced morphism $\rho: \mathfrak{M}' \rightarrow \mathfrak{M}$. Since ρ is a cohomology isomorphism, it is an isomorphism. Moreover,

$$0 \rightarrow \underline{\text{coker}} \rho_{n-1}^n \rightarrow W_n \rightarrow \underline{\text{ker}} \rho_n^{n+1} \rightarrow 0$$

by construction. But this is equal to $\underline{H}^n(\mathfrak{M}(n), \mathfrak{M}(n-1))$ as we can see from the long exact sequence

$$\begin{aligned} \cdots \rightarrow \underline{H}^n(\mathfrak{M}(n)) \rightarrow \underline{H}^n(\mathfrak{M}(n), \mathfrak{M}(n-1)) \rightarrow \underline{H}^{n+1}(\mathfrak{M}(n-1)) \\ \rightarrow \underline{H}^{n+1}(\mathfrak{M}(n)) \rightarrow \cdots \end{aligned}$$

For the proof of Propositions 5.4 and 5.5 we use ideas analogous to the ones of the nonequivariant case and the techniques of the construction used to prove Theorem 5.3. We content ourselves with an *outline of the proof of 5.5*: Let \mathfrak{M}_1 and \mathfrak{M}_2 be two minimal systems of DGA's for G such that \mathfrak{M}_2 is an elementary extension of \mathfrak{M}_1 with respect to d' : $W_n \rightarrow \underline{Z}^{n+1}(\mathfrak{M}_1)$. Let $f, g: \mathfrak{M}_2 \rightarrow \mathfrak{A}$ be two morphisms and $H_1: \mathfrak{M}_1 \rightarrow \mathfrak{A}(t, dt)$ a homotopy between $f|_{\mathfrak{M}_1}$ and $g|_{\mathfrak{M}_1}$. Then H_1

can be extended to a homotopy H_2 between f and g if and only if the morphism $\sigma_n: W_n \rightarrow \underline{Z}^{n+1}(\mathfrak{A})$ defined by

$$(7.5) \quad \sigma_n = f - g + (-1)^n \sum_{m=0}^M \frac{1}{m+1} k_m$$

is a coboundary; i.e., it factors through d via some $\sigma'_n: W_n \rightarrow \mathfrak{A}^n$; $k_m: W_n \rightarrow \mathfrak{A}^n$ is a morphism such that

$$H_1 \circ d' = \sum_{m=0}^M h_m t^m + \sum_{m=0}^M k_m t^m dt.$$

Then the homotopy H_2 is defined on W_n by

$$H_2 = f + (-1)^n \sum_{m=0}^M \frac{1}{m+1} k_m t^{m+1} - d \circ \sigma'$$

and, by the techniques of 5.10, it can be extended over \mathfrak{M}_2 .

We shall use these facts to prove Proposition 5.5 as follows. Suppose inductively that there is $f'_{n-1}: \mathfrak{M}(n-1) \rightarrow \mathfrak{A}$ such that the diagram

$$\begin{array}{ccc} & & \mathfrak{A} \\ & \nearrow f'_{n-1} & \downarrow \pi \\ \mathfrak{M}(n-1) & \xrightarrow{f} & \mathfrak{B} \end{array}$$

commutes up to homotopy. We want to extend it over $\mathfrak{M}(n)$, which is an elementary extension of $\mathfrak{M}(n-1)$ with respect to $d': W_n \rightarrow \underline{Z}^{n+1}(\mathfrak{M}(n-1))$. The morphism $f'_{n-1} \circ d'$ is a cocycle such that $\pi^*([f'_{n-1} \circ d']) = [f \circ d'] = [d \circ f] = 0$ in $H^{n+1}(\mathfrak{B}, W_n)$. Since π^* is an isomorphism, $[f'_{n-1} \circ d'] = 0$, i.e., $f'_{n-1} \circ d'$ factors through the differential of \mathfrak{A} via some $\tilde{f}_n: W_n \rightarrow \mathfrak{A}^{n+1}$. The obstruction to a homotopy between $f| \mathfrak{M}(n)$ and $\pi \circ f'_n$ is $[\sigma_n] \in H^{n+1}(\mathfrak{B}, W_n)$, where

$$\sigma_n = f| \mathfrak{M}(n) - \pi \circ \tilde{f}_n + (-1)^n \sum_{m=0}^M \frac{1}{m+1} k_m.$$

Let $[\tilde{\sigma}_n] \in H^{n+1}(\mathfrak{A}, W_n)$ be such that $\pi^*[\tilde{\sigma}_n] = [\sigma_n]$. Then $\pi \circ \tilde{\sigma}_n - \sigma_n = d \circ \sigma'_n$ (the difference is a coboundary). We correct \tilde{f}_n to $f'_n = \tilde{f}_n - \tilde{\sigma}_n$ and extend this over $\mathfrak{M}(n)$. The obstruction to a homotopy between $\pi \circ f'_n$ and $f| \mathfrak{M}(n)$ now vanishes.

8. Deferred proofs from §6.

PROOF OF LEMMA 6.3. First of all we have to determine the concept of equivariant transgression. Since

$$F^H = K(M(G/H), n) \rightarrow E^H \rightarrow X^H$$

is a transgressive fibration and since the transgression is functorial, there is a morphism $\tau: \underline{H}^n(F; \mathbf{Q}) \rightarrow \underline{H}^{n+1}(X; \mathbf{Q})$ such that the corresponding part of the following diagram commutes and, in particular, $p^* \circ \tau = 0$.

$$\begin{array}{ccccc}
 \underline{H}^n(F; \mathbf{Q}) & \longrightarrow & \underline{H}^{n+1}(E, F; \mathbf{Q}) & \longrightarrow & \underline{H}^{n+1}(E; \mathbf{Q}) \\
 \uparrow q & \searrow \tau & \uparrow & \nearrow p^* & \uparrow \\
 & & \underline{H}^{n+1}(X; \mathbf{Q}) & & \\
 & \searrow \tilde{q} & \uparrow & & \\
 & & \underline{Z}^{n+1}(\underline{\mathbb{G}}_X) & \xrightarrow{p^*} & \underline{Z}^{n+1}(\underline{\mathbb{G}}_E) \\
 \uparrow i & \searrow q' & \uparrow & & \uparrow d \\
 \ker q & & \underline{\mathbb{G}}_X^n & &
 \end{array}$$

Let $q: Q \rightarrow \underline{H}^n(F; \mathbf{Q})$ be an epimorphism, where Q is projective in Vec_G^* . Then there is a lifting q' of $\tau \circ q$. Moreover, $q' \circ i: \ker q \rightarrow \underline{Z}^{n+1}(\underline{\mathbb{G}}_X)$ is a morphism into the exact PL forms $\underline{B}^{n+1}(\underline{\mathbb{G}}_X) \subseteq \underline{Z}^{n+1}(\underline{\mathbb{G}}_X)$. We have the same situation as in the second diagram of the proof of 5.3. If we use the same method, we find morphisms \tilde{q} and r such that the entire diagram commutes.

Since $\rho: \mathfrak{A} \rightarrow \underline{\mathbb{G}}_X$ induces an isomorphism on cohomology and, in particular,

$$\rho^*: H^{n+1}(\mathfrak{A}; \underline{H}^n(F, \mathbf{Q})) \xrightarrow{\cong} H^{n+1}(\underline{\mathbb{G}}_X; \underline{H}^n(F; \mathbf{Q})),$$

there is a morphism $\sigma: \underline{H}^n(F; \mathbf{Q}) \rightarrow \underline{Z}^{n+1}(\mathfrak{A})$ such that $\rho^*[\sigma] = [\tilde{q}]$; then $\rho \circ \sigma = \tilde{q}$ after possible correction of \tilde{q} and r by a coboundary.

We consider the elementary extension $\tilde{\mathfrak{A}} = \mathfrak{A}(\underline{H}^n(F; \mathbf{Q}))$, where $d: \underline{H}^n(F; \mathbf{Q}) \rightarrow \underline{Z}^{n+1}(\mathfrak{A})$ is equal to σ and we define a morphism $\tilde{\rho}: \tilde{\mathfrak{A}} \rightarrow \underline{\mathbb{G}}_E$ such that $\tilde{\rho}|_{\mathfrak{A}} = \rho$ and $\tilde{\rho}|_{\underline{H}^n(F; \mathbf{Q})} = r$.

In order to prove that $\tilde{\rho}$ induces an isomorphism on cohomology, we consider the construction for each G/H , $H \subseteq G$, and apply the (nonequivariant) arguments of [3].

PROOF OF LEMMA 6.4. The long exact sequence (4.8) implies $\underline{H}^i(\tilde{\mathfrak{A}}, \mathfrak{A}) = 0$, $i < n$ and $\underline{H}^n(\tilde{\mathfrak{A}}, \mathfrak{A}) \cong W_n$. On the other hand, the spectral sequence (4.7) yields $H^n(\tilde{\mathfrak{A}}, \mathfrak{A}; W_n) \cong \text{Hom}(W_n, W_n)$. We construct a G -fibration $F = K(W_n^*, n) \rightarrow E \rightarrow X$ the equivariant k -invariant of which is the image of $[\text{id}] \in H^n(\tilde{\mathfrak{A}}, \mathfrak{A}; W_n)$ under

$$\delta^*: H^n(\tilde{\mathfrak{A}}, \mathfrak{A}; W_n) \rightarrow H^{n+1}(\mathfrak{A}; W_n) \cong H_G^{n+1}(X; W_n^*).$$

Since $(\text{id}, -\tilde{d}): W_n \rightarrow \tilde{\mathfrak{A}} \times \mathfrak{A}$ represents the cohomology class $[\text{id}]$, the morphism $-\tilde{d}: W_n \rightarrow \mathfrak{A}^{n+1}$ represents the k -invariant.

In order to show that $\tilde{\mathfrak{A}}$ is geometric for E we shall construct a system of DGA's $\hat{\mathfrak{A}}$ which is geometric for E (according to the previous lemma) and show that $\tilde{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$ are isomorphic. $\hat{\mathfrak{A}}$ is an elementary extension of \mathfrak{A} with respect to some morphism $\hat{d}: \underline{H}^n(F; \mathbf{Q}) \rightarrow \underline{Z}^{n+1}(\mathfrak{A})$ such that $\rho \circ \hat{d}$ induces the transgression $\tau: \underline{H}^n(F; \mathbf{Q}) \rightarrow \underline{H}^{n+1}(X; \mathbf{Q})$. Since $\underline{H}^n(F; \mathbf{Q}) = \underline{H}^n(K(W_n^*, n); \mathbf{Q}) = W_n$ (this follows from the nonequivariant case) we have

$$H^n(\hat{\mathfrak{A}}, \mathfrak{A}; W_n) \cong \text{Hom}(W_n, W_n) \cong \text{Hom}(W_n^*, W_n^*) = \text{Hom}(\pi_n(F), \pi_n(F)).$$

According to our discussion in (2.5), the equivariant k -invariant of the above G -fibration is the image of $[\text{id}]$ under

$$\delta^*: H^n(\mathfrak{X}, \mathfrak{A}; W_n) \cong \text{Hom}(\pi_n(F), \pi_n(F)) \rightarrow H^{n+1}(\mathfrak{A}, W_n) \cong H_G^{n+1}(X; \pi_n(F));$$

therefore it can be represented by $-\hat{d}$.

The morphisms \tilde{d} and \hat{d} differ by a coboundary $\tilde{d} - \hat{d} = d \circ \sigma$, $\sigma: W_n \rightarrow \mathfrak{A}^n$, because they represent the same cohomology class.

We define $A: \tilde{\mathfrak{X}} \rightarrow \hat{\mathfrak{X}}$ by $A|_{\mathfrak{A}} = \text{id}$ and $A|_{W_n} = \text{id} + \sigma$. This is obviously an isomorphism.

PROOF OF PROPOSITION 6.7. Our construction of the equivariant minimal model allows the transfer of the nonequivariant arguments to our case. We shall construct f inductively as a map between the equivariant Postnikov decomposition of X and Y . The other statements of the proposition are proved similarly.

Let $f_{n-1}: X_{n-1} \rightarrow Y_{n-1}$ be a G -map which induces $\tilde{f}| \mathfrak{N}(n-1): \mathfrak{N}(n-1) \rightarrow \mathfrak{N}(n-1)$. The first obstruction to lifting f_{n-1} to a G -map $h: X_n \rightarrow Y_n$ is the cohomology class $p_X^* f_{n-1}^*(k^{n+1}) \in H_G^{n+1}(X_{n-1}; \pi_n(Y))$.

$$\begin{array}{ccccc} X_n & \dashrightarrow & Y_n & \xrightarrow{\quad} & P \\ p_X \downarrow & & \downarrow p_Y & & \downarrow \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} & \xrightarrow{k^{n+1}} & K(\pi_n(Y), n+1) \end{array}$$

But $p_X^* f_{n-1}^*(k^{n+1}) = \tilde{f}^* p_Y^*(k^{n+1}) = \tilde{f}^*(0) = 0$ in $H^{n+1}(\mathfrak{N}(n); \pi_n(Y))$.

Let $\tilde{h}: \mathfrak{N}(n) \rightarrow \mathfrak{N}(n)$ be a morphism induced by h such that $\tilde{h}| \mathfrak{N}(n-1) = \tilde{f}| \mathfrak{N}(n-1)$. The morphisms \tilde{h} and \tilde{f} need not be homotopic on $\mathfrak{N}(n)$. We consider the difference

$$\Delta: V_n \rightarrow \mathfrak{N}(n), \quad \Delta = \tilde{h}| V_n - \tilde{f}| V_n,$$

where $\mathfrak{N}(n) = \mathfrak{N}(n-1)(V_n)$, and the cohomology class it represents. According to the classification theorem (2.4), we can change the lifting h by $[\Delta]$ to a lifting $f_n: X_n \rightarrow Y_n$. Then the new Δ is a coboundary and thus, by the discussion in the proof of 5.5, \tilde{f}_n and $\tilde{f}| \mathfrak{N}(n)$ are homotopic.

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